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On the connection between financial processes with stochastic volatility and nonextensive statistical mechanics

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Abstract. The $GARCH$ algorithm is the most renowned generalisation of Engle's original proposal for modelising *returns*, the $ARCH$ process. Both cases are characterised by presenting a time dependent and correlated variance or *volatility*. Besides a memory parameter, b, (present in $ARCH$) and an independent and identically distributed noise, ω, $GARCH$ involves another parameter, c, such that, for $c = 0$, the standard $ARCH$ process is reproduced. In this manuscript we use a generalised noise following a distribution characterised by an index q_n, such that $q_n = 1$ recovers the Gaussian distribution. Matching low statistical moments of $GARCH$ distribution for returns with a q-Gaussian distribution obtained through maximising the entropy $S_q = \frac{1-\sum_i p_i^q}{q-1}$, basis of nonextensive statistical mechanics, we obtain a sole analytical connection between q and (b, c, q_n) which turns out to be remarkably good when compared with computational simulations. With this result we also derive an analytical approximation for the stationary distribution for the (squared) volatility. Using a generalised Kullback-Leibler relative entropy form based on S_q, we also analyse the degree of dependence between successive returns, z_t and z_{t+1}, of $GARCH(1,1)$ processes. This degree of dependence is quantified by an entropic index, q^{op}. Our analysis points the existence of a unique relation between the three entropic indexes q^{op}, q and q_n of the problem, independent of the value of (b, c).

PACS. 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 89.65.Gh Economics; econophysics, financial markets, business and management

1 Introduction

The study of time series plays an important role in science due to their ubiquity in both natural and artificial systems. They can be found, e.g., in geoseismic (earthquakes), meteorological (El Niño), physiological (electroencephalographic profiles) or financial phenomena [1–5]. They comprise a sequentially ordered set of random variables, correlated or not, following a certain probability function. For the uncorrelated case, the most perceptive is to consider the time series as a succession of values that are associated to the same probability distribution, like it occurs for the ordinary random walk, where probability of a certain jump value is constant in time. This kind of process is defined as *homoskedastic*. However, there are phenomena for which the probability distribution associated to the stochastic variable at some time step t depends explicitly on t and these are named *heteroskedastic*. A simple way to obtain a heteroskedastic process is to consider the same probability functional for all times, but with a varying second-order moment (or width). In this sort of stochasticity we can include financial time series, namely return time series, where second-order moment time dependence is a feature more than well-known [6]. Aiming to mimic this type of systems, Engle introduced in 1982 the autoregressive conditional heteroskedasticity ($ARCH(s)$) (s will be defined later on) process [7] which is considered a landmark in finance, comparable to the Black-Scholes equation [8], because of its wide use [9,10]. Albeit its extraordinary importance, Engle's model can, in many applications, reach large values of parameter s which carries out implementation difficulties. This point inspired T. Bollerslev to generalise it defining the $GARCH(s,r)$ [11] (G stands for *generalised*) which presents a more flexible structure and enabling that previous data, only correctly mimiced with large s, could be reproduced with simple $GARCH(1,1)$ process. Due to its financial cradle these models are not well-known in physics, nevertheless they can be very useful to illustrate many traditional physical problems (see, e.g., [12]). In this article we give sequence to the ansatz presented by us in a previous work [13] for the $ARCH\,(1)$

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process, generalising it to the $GARCH(1,1)$. Moreover, we detail the physical justification for why these models (which present time dependent variance) accommodate well within the current nonextensive statistical mechanical theory. We also present an analytical form for the distribution for the (squared) volatility and analyse the degree of dependence between successive *returns*. The manuscript is organised as follows: in Section 2 we introduce the $GARCH(s,r)$ process and some of its properties; In Section 3.1 we present our connection between $GARCH(1,1)$ and nonextensive entropy. In addition, along the lines of superstatistics [18], we derive the distribution for the (squared) volatility in Section 3.2. In Section 4, applying a generalised Kullback-Leibler relative entropy we analyse the degree of dependence between z_t and z_{t+1} elements of a $GARCH(1,1)$ time series and its relation with non-Gaussianity. Some final comments are made in Section 5.

2 The GARCH model

As settled by Engle [7], we will define an autoregressive conditional heteroskedastic time series as a discrete time stochastic process, z_t,

$$z_t = \sigma_t \, \omega_t, \tag{1}$$

where ω_t is an independent and identically distributed random variable with null mean and unitary variance, i.e., $\langle \omega_t \rangle = 0$ and $\langle \omega_t^2 \rangle = 1$. The quantity σ_t, named *volatility* is time varying, positive defined and dependent of the past values of the *return* z_t. According to its definition, the process presents mean zero, is uncorrelated ($\langle z_t z_{t'} \rangle \sim \delta_{tt'}$) and has a conditional variance, σ_t^2, that evolves in time.

In his original paper Engle [7] suggests a possible expression for σ_t^2 defining it as a linear function of past squared values of z_t known as $ARCH(s)$ *linear process*,

$$\sigma_t^2 = a + \sum_{i=1}^{s} b_i \, z_{t-i}^2, \qquad (a, b_i \geqslant 0). \tag{2}$$

With the aim of solving the weak points of the $ARCH$ process that were referred in Section 1, it was introduced the linear $GARCH(s,r)$ which presents a more flexible structure for the functional form of σ_t^2 decreasing the direct influence of z on σ_t^2 (for details see [11]):

$$\sigma_t^2 = a + \sum_{i=1}^{s} b_i \, z_{t-i}^2 + \sum_{i=1}^{r} c_i \, \sigma_{t-i}^2, \qquad (a, b_i, c_i \geqslant 0). \tag{3}$$

Like its predecessor, the $GARCH(s,r)$ model also captures the recognised tendency for *volatility clustering* (evident in financial time series) and is very similar to intermittent fluctuations in turbulent flows [14]: large(small) values of z_t are usually followed by large(small) values. However, due to the *arbitrary sign* of ω_t, it can be verified that, although $\langle z_t z_{t'} \rangle \sim \delta_{tt'}$, $\langle |z_t| \; |z_{t'}| \rangle$ is *not* proportional to $\delta_{tt'}$. As a matter of fact it can be verified for $GARCH(1,1)$ [11] that the covariance of z_t^2,

$$\mathrm{cov}\left(z_t^2, z_{t'}^2\right) \equiv \left\langle z_t^2 \, z_{t'}^2 \right\rangle - \left\langle z_t^2 \right\rangle \left\langle z_{t'}^2 \right\rangle,$$

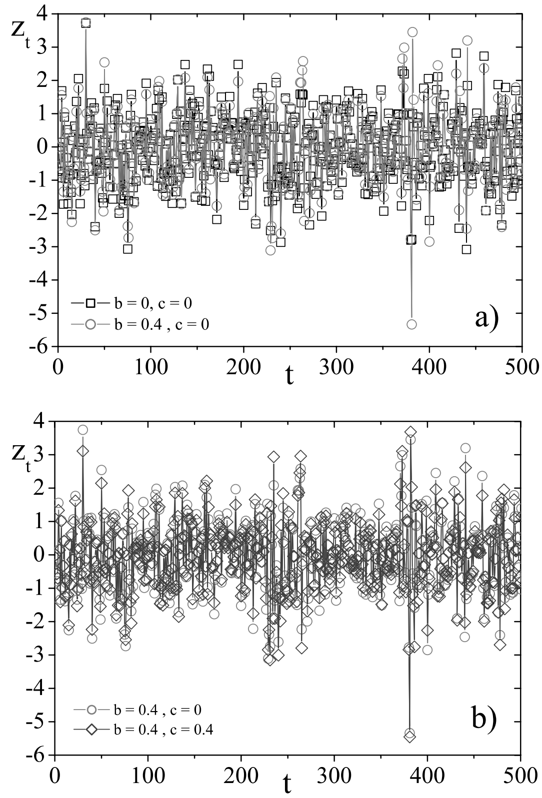

Fig. 1. Typical $GARCH\,(1,1)$ time series obtained for a Gaussian noise, $q_n = 1$. In (a) we present time series which correspond to pure Gaussian (□) and $ARCH\,(1)$ (∘). In (b) the introduction of parameter c (⋄) increases the probability for larger values of $|z_t|$, thus increasing tails in $P\,(z)$.

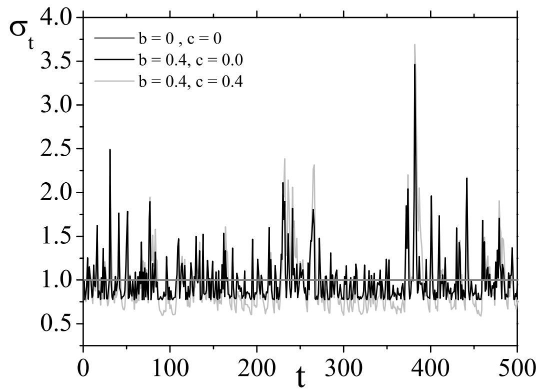

Fig. 2. Time dependence of volatility σ for the $GARCH$ processes presented in Figure 1. Here is clearly visible the difference between $c = 0$ ($ARCH$) and $c \neq 0$ ($GARCH$). For the same values of b we are able to obtain greater values of σ, thus leading to fatter tails in $P\,(z)$.

decreases as an exponential law with characteristic time $\tau \equiv |\ln(b_1 + c_1)|^{-1}$, which, unfortunately, is not in accordance to what is empirically verified in financial time series [15]. For $c_i = 0\;(\forall i)$, $GARCH(s,r)$ straightforwardly reduces to the linear $ARCH(s)$. See Figures 1, 2 and 3 for typical realisations.

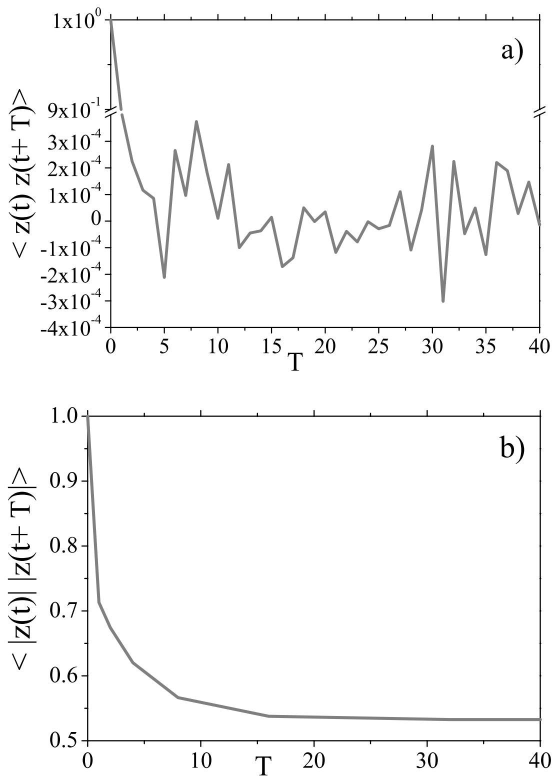

Fig. 3. In (a) correlation between returns $z(t+T)$ and $z(t)$ vs. time lag T . All values, except for $T=0$, are at noise level. In (b) correlation between absolute values of returns which present a decay of time lag T. The run analysed was the same presented in Figure 1.

Let us continue to focus on the simplest and most used of the $GARCH$ processes, the $GARCH(1,1)$, as it is found to be sufficient to mimic the majority of the applications. In this case,

$$\sigma_t^2 = a + b\, z_{t-1}^2 + c\, \sigma_{t-1}^2, \tag{4}$$

so that the process is completely defined when a, b, c, and the noise nature, $P_n(\omega_t)$, are specified.

Combining equations (1) and (4) we obtain the nth order moment for the stationary $P(z)$ distribution, particularly the second,

$$\bar{\sigma}^2 \equiv \left\langle z_t^2 \right\rangle = \left\langle \sigma_t^2 \right\rangle = \frac{a}{1-b-c}, \qquad (b+c<1), \tag{5}$$

and the fourth moment,

$$\left\langle z_t^4 \right\rangle = a^2 \left\langle \omega_t^4 \right\rangle \frac{1+b+c}{(1-b-c)\left(1-2\,b\,c-c^2-b^2\left\langle \omega_t^4 \right\rangle\right)}. \tag{6}$$

The condition $b+c<1$ is actually important, since it guarantees that the $GARCH(1,1)$ corresponds exactly to an infinite-order ARCH process [16]. Furthermore, for Gaussian noise ($q_n = 1$), together with the condition that c is larger than approximately 0.7, the previous condition not only assures finiteness for second and fourth moments, but also leads to a significant increase in the covariance time scale, τ, when compared with $ARCH(1)$ (for details consult [6]).

Let us now assume, for simplicity and without lack of generality, a $GARCH(1,1)$ process that generates time series with unitary variance, i.e., $\bar{\sigma}^2 = 1$, which imposes $a = 1-b-c$. Now, for this process, the fourth moment is numerically equal to the kurtosis $k_x \equiv \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2}$, and thus we get,

$$\left\langle z_t^4 \right\rangle = k_z = k_\omega \left(1 + b^2 \frac{k_w - 1}{1-c^2-2\,b\,c-b^2\,k_\omega}\right), \tag{7}$$

where $c^2 + 2\,b\,c + b^2\,k_\omega < 1$. It is clear from equation (7) that $GARCH(1,1)$ generates distributions $P(z)$ with fatter tails than those of the noise ω_t, and fatter also than the ones obtained with $ARCH(1)$ with the same b [19]. It is also possible to see that parameter c is only useful for $b \neq 0$, otherwise $GARCH(1,1)$ process reduces to constructing stationary probability distributions with a kurtosis k_z equal to k_w.

3 The ansatz connecting GARCH and nonextensive statistical mechanics

3.1 Stationary distribution for returns

We shall now establish a connection — physically motivated in the next subsection — between the present stochastic process and the current nonextensive statistical mechanical framework, based on the entropic form [20],

$$S_q = \frac{1-\int_{-\infty}^{+\infty} [p(z)]^q\, dz}{q-1}, \qquad (q \in \Re). \tag{8}$$

This entropy is currently referred to as *nonextensive* because it is so for *independent* subsystems. It can however be *extensive* in the presence of scale-invariant correlations [21–24]. The associated statistics has been successfully applied to phenomena presenting some kind of scale-invariant geometry, like in low-dimensional dissipative and conservative maps [25], anomalous (correlated) diffusion [26], turbulent flows [27], Langevin dynamics with fluctuating temperature [18,29,30], long-range many-body classical Hamiltonians [31], among many others [32]. Entropy (8) constitutes a generalisation of the Boltzmann-Gibbs (BG) one, namely $S_{BG} = -\int_{-\infty}^{+\infty} p(z)\, \ln p(z)\, dz$. Indeed, this celebrated expression is recovered as the $q \to 1$ limit of entropy (8).

By applying the standard variational principle on entropy (8) with the constraints $\int_{-\infty}^{+\infty} p(z)\, dz = 1$ and

$$\int_{-\infty}^{+\infty} z^2\, [p(z)]^q\, dz \Big/ \int_{-\infty}^{+\infty} [p(z)]^q\, dz = \bar{\sigma}_q^2$$

[33–35] ($\bar{\sigma}_q^2$ is defined as the *generalised second-order moment*) we obtain

$$p(z) = \frac{\mathcal{A}}{\left[1+\mathcal{B}\,(q-1)\,z^2\right]^{\frac{1}{q-1}}}, \qquad \left(q < \frac{5}{3}\right), \tag{9}$$

where

$$\mathcal{B} \equiv \frac{1}{\bar{\sigma}^2\ (5 - 3\,q)}, \tag{10}$$

$$\bar{\sigma}^2 \equiv \int_{-\infty}^{+\infty} z^2\, p(z)\, dz, \tag{11}$$

and

$$\mathcal{A} = \frac{\Gamma\left[\frac{1}{q-1}\right]}{\sqrt{\pi}\,\Gamma\left[\frac{3-q}{2q-2}\right]}\sqrt{(q-1)\,\mathcal{B}}. \tag{12}$$

Standard and generalised second-order moments, $\bar{\sigma}^2$ and $\bar{\sigma}_q^2$, are related by $\bar{\sigma}^2 = \bar{\sigma}_q^2 \frac{3-q}{5-3\,q}$. In the limit $q \to 1$, distribution (9) becomes a Gaussian. If $q = \frac{3+m}{1+m}$ $(m = 1, 2, 3...)$, distribution (9) recovers the Student's t-distribution with m degrees of freedom; if $q = \frac{n-4}{n-2}$ $(n = 3, 4, 5...)$, it recovers the so-called r-distribution with n degrees of freedom [36].

Defining the *q-exponential* function as,

$$\exp_q(x) \equiv [1 + (1-q)x]^{\frac{1}{1-q}} \quad (\exp_1(x) = e^x), \tag{13}$$

we can rewrite the above distribution as follows,

$$p(z) = \mathcal{A}\, e_q^{-\,\mathcal{B}\,z^2}, \tag{14}$$

from now on referred to as *q-Gaussian*. The fourth moment of $p(z)$ is,

$$\left\langle z^4 \right\rangle = 3\,\left(\bar{\sigma}^2\right)^2 \frac{3\,q-5}{5\,q-7}. \tag{15}$$

Let us now propose the ansatz $p(z) \simeq P(z)$ with $\bar{\sigma}^2 = 1$ [13]. Specifically, we will impose the matching of equations (7) and (15). Again, we assume that the noise ω_t follows the generalised distribution

$$P_n(\omega) = \frac{\mathcal{A}_{q_n}}{\left[1 + \frac{q_n-1}{5-3q_n}\,\omega^2\right]^{\frac{1}{q_n-1}}}, \qquad \left(q_n < \frac{5}{3}\right), \tag{16}$$

defined by the index q_n; its variance equals unity, and $\mathcal{A}_{q_n}$ is uniquely determined through normalization. We are then able to establish a relation between parameters b, c and indices q_n and q:

$$b = \frac{\sqrt{(q-q_n)(\,f(q_n,q) - c^2\, f(q,q_n))}}{f(q_n,q)} - \frac{c\,(q-q_n)}{f(q_n,q)}, \tag{17}$$

with $f(x, y) = (5 - 3x)(2 - y)$. For $b = c = 0$ we have $q = q_n$. Naturally, the connection indicated in equation (17) and illustrated in Figure 4 for typical values of q_n reduces, for $c = 0$, to the one obtained [13] for the $ARCH\,(1)$ model (linear finite-order $ARCH$ process), namely

$$b = \sqrt{(q-q_n)/[(5-3q_n)(2-q)]},$$

hence

$$q = \left[q_n + 2b^2(5-3q_n)\right] / \left[1 + b^2(5-3q_n)\right].$$

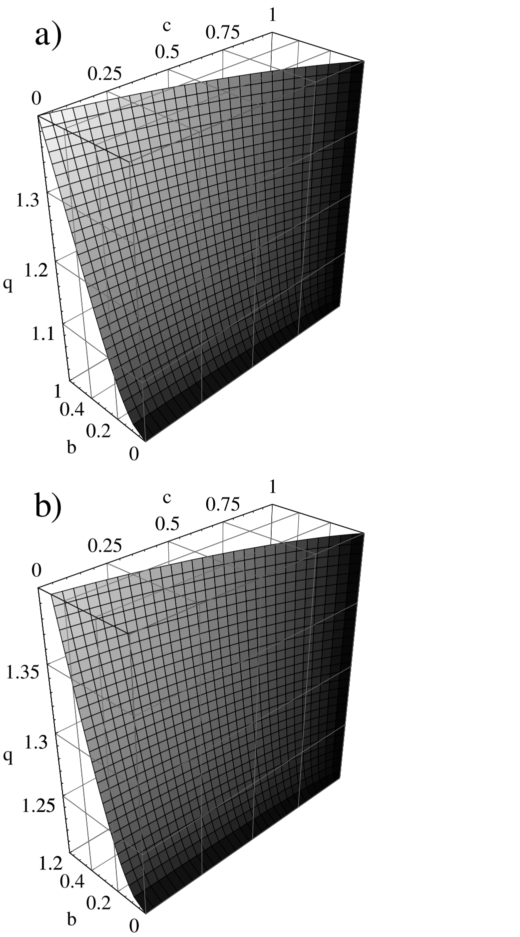

Fig. 4. Diagram (q, b, c) for $q_n = 1$ in (a) and $q_n = 1.2$ in (b). In (b) we can see that the greatest allowed value of b $(c = 0)$ is $b = \frac{1}{\sqrt{4.2}} \simeq 0.488$.

To verify the above ansatz we generated, for typical values of (b, c, q_n) and using an algorithm based on equations (1) and (4), a set of $GARCH$ time series. Then we computed the corresponding probability density functions and compared them with the histograms (with any adequately chosen interval δ) associated with the q-Gaussian distribution with q satisfying the ansatz. We compared then the numerical probability density functions (PDFs) with $H(z)$

$$H(z) = \int_{z\,+\,\delta/2}^{z\,+\,\delta/2} p\,(x)\;dx = \frac{\Gamma\left[\frac{1}{q-1}\right]}{2\,\Gamma\left[\dfrac{1}{q-1} - \dfrac{1}{2}\right]}\sqrt{\frac{1-q}{\pi\,(\,3\,q-5)}}$$

$$\times \left\{ (\delta - 2\,z)\ {}_2F_1\left(\frac{1}{2}, \frac{1}{q-1}; \frac{3}{2}; \frac{(q-1)\,(\delta-2\,z)^2}{4\,(3\,q-5)}\right) \right.$$

$$\left. + (\delta + 2\,z)\ {}_2F_1\left(\frac{1}{2}, \frac{1}{q-1}; \frac{3}{2}; \frac{(q-1)\,(\delta+2\,z)^2}{4\,(3\,q-5)}\right) \right\}, \tag{18}$$

where ${}_2F_1$ is the hypergeometric function. As can be seen in Figures 5 and 6, the accordance between stationary PDFs and the PDFs obtained by using, in equation (9),

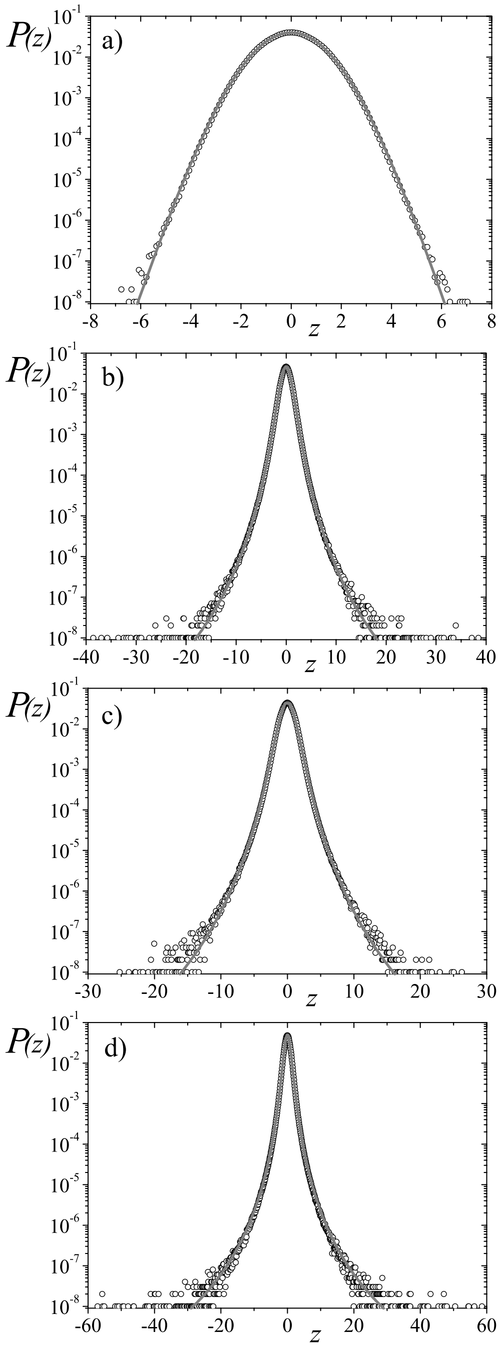

Fig. 5. PDFs for a $q_n = 1$ noise and typical values of (b; c) pair. (a) (0.1; 0.1), $q = 1.021$ ($\chi^2 = 2.35 \times 10^{-9}$); (b) (0.1; 0.88), $q = 1.287$ ($\chi^2 = 4.59 \times 10^{-10}$); (c) (0.4; 0.1), $q = 1.26$ ($\chi^2 = 2.44 \times 10^{-9}$); (d) (0.4; 0.4), $q = 1.38$ ($\chi^2 = 3.22 \times 10^{-7}$).

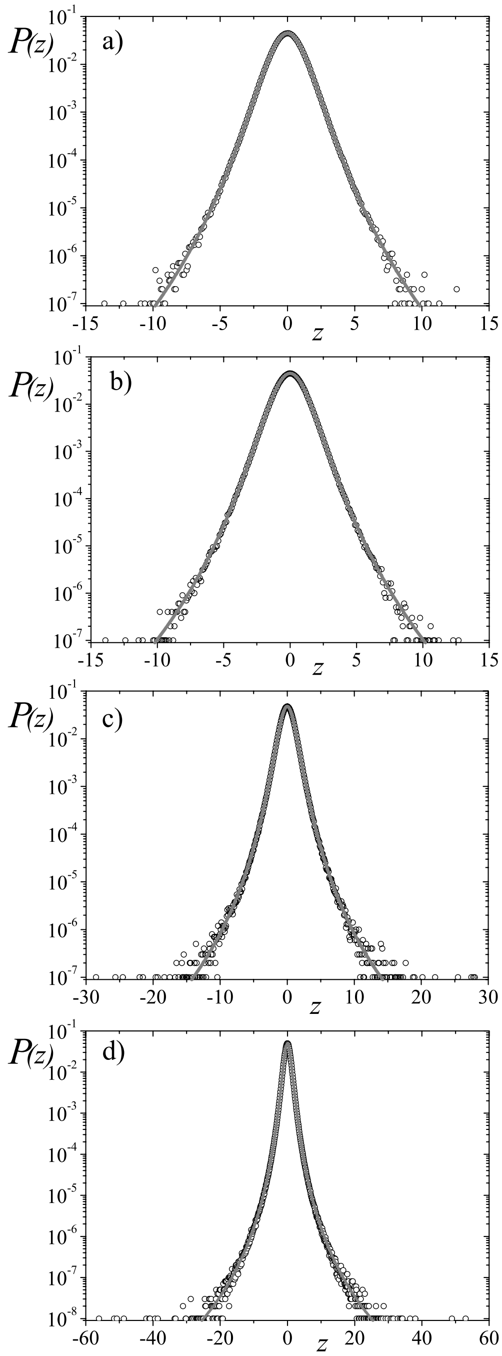

Fig. 6. PDFs for a $q_n = 1.2$ noise and typical values of (b; c) pair. (a) (0.1; 0.1), $q = 1.211$ ($\chi^2 = 8.67 \times 10^{-10}$); (b) (0.1; 0.5), $q = 1.221$ ($\chi^2 = 6.01 \times 10^{-10}$); (c) (0.3; 0.25), $q = 1.310$ ($\chi^2 = 8.11 \times 10^{-9}$); (d) (0.3; 0.45), $q = 1.35$ ($\chi^2 = 7.36 \times 10^{-9}$).

the value of q satisfying equation (17), is quite satisfactory. The values of b and c used in panels of Figures 5 and 6 were chosen aiming to give a general view of the quality of our propose throughout the domain which in b and c are allowed to vary keeping finite variance and kurtosis. In their captions we present also the values of the χ^2 error function,

$$\chi^2 \equiv \frac{1}{N} \sum_{i=1}^{N} \left[P\left(z\right) - H\left(z\right) \right]^2 .$$

Another way to evaluate the slight discrepancy between $P(z)$ and $p(z)$ (or $H(z)$) is to compute the percentual error in the sixth-order moment between PDFs. The results presented in Table 1 show that discrepancies are never larger than 3%, which in practice can be considered negligible. It is interesting to notice the remarkable agreement at least down to $p(z) = 10^{-6}$ (typical limit used in finance, see reference [4]).

Table 1. Percentual error in the *sixth*-order moment between numerical and ansatz PDFs presented in Figures 5 and 6.

q_n	b	c	$\langle z^6 \rangle_{\text{numerical}}$	$\langle z^6 \rangle_{\text{ansatz}}$	error (%)
1	0.1	0.2	13.97	14.05	0.60
1	0.1	0.88	66.75	65.46	1.93
1	0.4	0.1	61.45	62.33	1.44
1	0.4	0.4	591.91	517.67	1.75
1.2	0.1	0.1	49.41	49.08	0.67
1.2	0.1	0.5	55.93	55.43	0.89
1.2	0.3	0.25	181.48	179.44	1.13
1.2	0.3	0.45	1416.41	1455.37	2.75

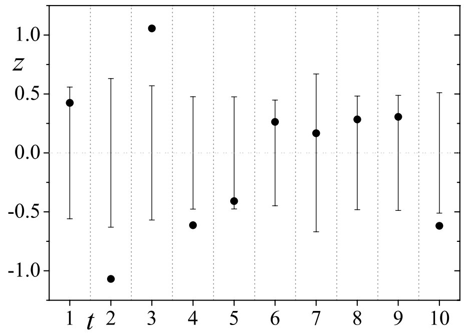

Fig. 7. Superstatistical illustration for the the 10 first time steps of a *GARCH* process with $a = b = 0.4$ and Gaussian noise. For this case, each time step (or cell) is characterised by a certain width (represented by the full line) of the Gaussian which is associated to the obtained value of z (full circles).

3.2 Stationary distribution for (squared) volatility

The good results of the proposal presented in the previous sub-section have their explanation in the main feature of the ARCH class of processes, namely the temporal dependence of the variance [28]. This fact allow us to think of them along the same lines of Wilk and Włodarczyk [29] and of Beck [30] which led to the *superstatistics* (statistics of statistics) recently advanced by Beck and Cohen [18]. This approach was developed to treat driven non-equilibrium systems composed, for instance, of smaller cells *in local equilibrium*, thus obeying BG statistics with a distribution $P_{BG} \propto e^{-\beta E}$, E being the energy. The long-term stationary state presents a spatio/temporary fluctuating temperature following a distribution $f(\beta)$. In the long-term, the probability density function for the nonequilibrium system comes from BG statistics associated with the small cells that are averaged over the various β, i.e.,

$$P_{\text{stationary}}(E) = \int f(\beta)\; P_{BG}(E)\, d\beta. \tag{19}$$

For long times, we can study *GARCH* as a stationary diffusion process, composed by t increments (or cells), each of them with a certain distribution width, σ_t for variable z_t (see illustration in Fig. 7). So, instead of defining the intensive fluctuating parameter, β, as the inverse temperature, we will define it as an inverse second-order moment, $\beta_\sigma = \frac{1}{2\sigma^2}$. In other words, for each "cell", variable z follows a certain $q_n - Gaussian$ conditioned to an instantaneous second-order moment, σ_t^2, which is associated with a probability distribution, $p_\sigma(\sigma^2)$. The stationary probability distribution $p(z)$ is thus given by

$$p(z) = \int_0^\infty p_\sigma(\sigma^2)\; p(z|\sigma^2)\; d(\sigma^2), \tag{20}$$

where $p(z|\sigma^2)$ is the *conditional* probability of having a value z for the return given a value σ^2. The homoskedastic case corresponds to $p_\sigma\left(\sigma^2\right) = \delta\left(\sigma^2 - \bar{\sigma}^2\right)$. Let us focus, for now, on the case $q_n = 1$ (Gaussian noise) with

$$p(z|\sigma^2) = \frac{1}{\sqrt{2\,\pi\,\sigma^2}}\, e^{-\frac{z^2}{2\,\sigma^2}}. \tag{21}$$

In their proposal Beck and Cohen [18] showed that, if the intensive parameter β is associated with a Gamma distribution, then the macroscopic non-equilibrium steady state follows *exactly* a $q-$*exponential* distribution (see also [29,30]) when the intensive parameter follows a Gamma distribution. In other words,

$$e_q^{-\beta'\,E(z)} = \int \frac{e^{-\beta/b}}{b\,\Gamma(c)} \left(\frac{\beta}{c}\right)^{c-1} e^{-\beta\,E(z)}\, d\beta.$$

Following the reverse line and assuming

$$p(z) = P_{\text{stationary}} \simeq P(z),$$

we are lead to a Gamma distribution in β_σ or to the following distribution in $\left(\sigma^2\right)$,

$$p_\sigma\left(\sigma^2\right) = \frac{\exp\left(-\frac{1}{2\kappa\sigma^2}\right)\left(\sigma^2\right)^{-1-\lambda}}{(2\kappa)^\lambda\;\Gamma\left[\lambda\right]}, \tag{22}$$

where

$$\lambda = \frac{1}{q-1} - \frac{1}{2} \tag{23}$$

and

$$\kappa = \frac{1-q}{\bar{\sigma}^2\,(3\,q-5)}. \tag{24}$$

It is interesting to mention that distribution (22), often called *inverted Gamma distribution*, is also the stationary solution for the Fokker-Planck equation obtained from stochastic differential equations with multiplicative noise like (in Itô notation),

$$dx = -\gamma\,(x-\theta)\,dt + \kappa\,x\,dW_t,$$

which is used in stochastic volatility models too, but obviously in a continuous approach [38]. This resemblance is in complete agreement with the discrete multiplicative noise structure for σ_t^2 recurrence equation (3) and other kinds of equations [39]. As can be seen in Figure 8, the ansatz gives also a quite satisfactory description for the probability distribution in the (squared) volatility, corroborating the

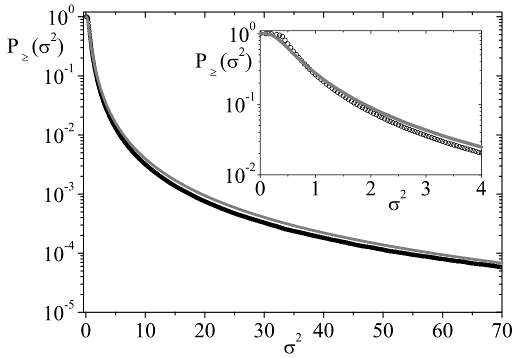

Fig. 8. The symbols in black represent the cumulative probability distribution, $P_\geqslant(\sigma^2)$ obtained from numerical simulation for a Gaussian noise with $b = c = 0.4$. The gray line represents the same distribution as given by equation (22) with $(\kappa, \lambda, \bar{\sigma}^2) = (0.444, 2.125, 1)$ satisfying equations (23) and (24).

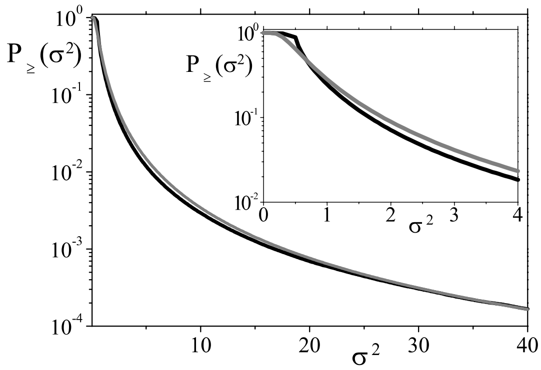

Fig. 9. The line in black represent the cumulative probability distribution, $P_\geqslant(\sigma^2)$ obtained from numerical simulation for a q_n-Gaussian noise ($q_n = 1.15$) with $(b, c) = (0.5, 0)$. The gray line represents the same distribution as given by equation (26) with $(\kappa, \lambda, \bar{\sigma}^2) = (0.365, 2.371, 1)$, and $q_\sigma = 1$.

connection between the ARCH class of processes, nonextensive statistical mechanics, and superstatistics. The analytic expressions (22), (23) and (24) can be very useful in applications related, among others, to option prices [4,37] where volatility forecasting plays a particularly important role [40,41].

For the case of a q_n-Gaussian noise (with $q_n > 1$) similar arguments can be used. However, the achievement of an analytical solution for equation (20) which generalises the approach above is not a trivial task.

For the conditional probability $p(z\,|\sigma^2)$ we know a satisfactory answer [42], namely

$$p(z|\sigma^2) = \frac{\mathcal{A}_{(q_n,\sigma^2)}}{\left[1 + \mathcal{B}_{(q_n,\sigma^2)}\ (q_n - 1)\ z^2\right]^{\frac{1}{q_n-1}}}. \quad (25)$$

But the same does not happen with $p_\sigma\left(\sigma^2\right)$. Assuming a continuous approach in q_n, a good ansatz for describing $p_\sigma\left(\sigma^2\right)$ is

$$p_\sigma\left(\sigma^2\right) \propto \left(\sigma^2\right)^{-1-\lambda} \exp_{q_\sigma}\left(-\frac{1}{2\,\kappa\,\sigma^2}\right) = \left(\sigma^2\right)^{-1-\lambda}\left(1 + \frac{q_\sigma - 1}{2\,\kappa}\frac{1}{\sigma^2}\right)^{\frac{1}{1-q_\sigma}}, \quad (26)$$

where q_σ is an index which depends on q_n, i.e., $q_\sigma(q_n)$ such that $q_\sigma(1) = 1$. For large σ^2,

$$p_\sigma\left(\sigma^2\right) \sim \left(\sigma^2\right)^{-1-\lambda}. \quad (27)$$

The integral $\int_0^\infty \sigma^2\, p_\sigma(\sigma^2)\, d(\sigma^2)$ should equal the mean value $\bar{\sigma}^2 = \frac{a}{1-b-c}$. This yields,

$$\frac{1 + \lambda - \lambda\, q_\sigma}{2\,\kappa\,(\lambda - 1)} = \bar{\sigma}^2, \quad (28)$$

and

$$\int_{-\infty}^{\infty} z^4 \int_0^\infty p_\sigma\left(\sigma^2\right)\ p\left(z\|\sigma^2\right)\ d\left(\sigma^2\right) dz = 3\left(\bar{\sigma}^2\right)^2 \frac{3\,q - 5}{5\,q - 7}. \quad (29)$$

From equation (27) and by adjusting numerically the curves for the cumulated probability distributions, $P_\geqslant(\sigma^2)$, we were able to determine λ, and then q_σ and κ from equations (28) and (29). The procedure appears to be valid for values of q_n close to unity. From the analysis of some values of q_n we verified that the value of q_σ equals 1 for every q_n considered. Figures 8 and 9 confirm that our proposal produces a satisfactory approximation when compared with numerical simulations, particularly for large values of volatility (which are, in turn, responsible for the large returns). For the q_n-Gaussian noise case, although some discrepancy exits for small σ^2, the tail is remarkably good.

4 Degree of dependence between successive returns

As stated in Section 2, stochastic variables, $\{z_t\}$, in a $GARCH$ process are uncorrelated (see Eq. (1)). However, if we combine equations (1) and (3), we can verify that they are not independent. More specifically, for $GARCH(1,1)$ ($\bar{\sigma}^2 = 1$), we have

$$z_t = \sqrt{1 + b\left(z_{t-1}^2 - 1\right) + c\left(\frac{z_{t-1}^2}{\omega_{t-1}^2} - 1\right)}\,\omega_t. \quad (30)$$

One of the possible measures of the dependence between the $(z_t; z_{t-1})$ stochastic variables consists in using a generalised form of the Kullback-Leibler relative entropy [43], namely

$$I_{q'}(p_1, p_2) \equiv -\int p_1(u)\, \ln_{q'}\left[\frac{p_2(u)}{p_1(u)}\right] du, \quad (31)$$

where $\ln_{q'}(x) \equiv \frac{x^{1-q'}-1}{1-q'}$ (q'-logarithm). In the limit $q' \to 1$, $I_{q'}$ recovers the standard Kullback-Leibler form [44]. This generalised relative entropy equals zero whenever $p_2(u) = p_1(u)$, and has the same sign as q' otherwise. With these properties we can use $I_{q'}$ (with $q' > 0$) as a way to compute the "distance", in probability space, from $p_2(u)$ to $p_1(u)$. Assume that u is a two dimensional random variable $u \equiv (x, y)$ so that $p_1(x, y)$ represents the *joint distribution* of (x, y), and $p_2(x, y) \equiv h_1(x)\, h_2(y)$, where $h_1(x) \equiv \int p_1(x, y)\, dy$ and $h_2(y) \equiv \int p_1(x, y)\, dx$ are the *marginal distributions*. Random variables x and y are independent if $p_1(x, y) = p_2(x, y)$. With the functional $I_{q'}$ we can measure the degree of dependence *via* distance between probabilities $p_1(x, y)$ and $p_2(x, y)$. For this case, $u \equiv (x, y)$ and $q' > 0$, it was shown [46] that, besides a lower bound $I_{q'} = 0$ (total independence of x and y), $I_{q'}$ presents, for every value of q', an upper bound (complete dependence between x and y) given by,

$$I_{q'}^{\mathrm{MAX}}(p_1, p_2) = -\int\int [p_1(x, y)]^{q'} \{\ln_{q'} h_1(x) + (1 - q')[\ln_{q'} h_1(x)][\ln_{q'} h_2(y)]\}\, dx\, dy. \quad (32)$$

Dividing equations (31) by (32) we define

$$R_{q'} \equiv \frac{I_{q'}}{I_{q'}^{\mathrm{MAX}}} \quad \in [0, 1]. \quad (33)$$

This ratio can be used as a criterion for measuring the degree of dependence between random variables. Indeed, it presents an optimal q', noted q^{op} [45], for which the sensitivity of $R_{q'}$ is maximal (q^{op} corresponds to the inflexion point of $R_{q'}(q')$). Higher (lower) values of q^{op} represents lower (higher) degree of dependence [46]. Taking $x = z_t$ and $y = z_{t-1}$ generated from equation (30), and applying equation (33), we obtained the curves presented in Figure 10 for typical values of (b, c, q_n). For each set we determined q^{op} and plotted it versus q obtained from equation (17): see Figure 11. For both noises that have been illustrated, q^{op} monotonically decreases with q. For fixed q_n, this (decreasing) curve does not depend on the values of (b, c). An illustration of this independence is indicated in Figure 11 by using different pairs (b, c) that give the same q. This fact suggests the existence of a relation between non-Gaussianity (represented by q), the degree of dependence quantified with q^{op}, and the noise index q_n. This triangular relation (q,q^{op},q_n) is analogous to another one which could relate the sensitivity, relaxation and stationarity in weakly chaotic systems such as those in which long-range interactions are assumed [47].

Let us now compare these results with others obtained by empirical analysis of index returns time series [48]. Invoking Drost and Nijman [49] work on the temporal aggregation of $GARCH$ processes, we can state that a $GARCH$ process with value q can be interpreted as the result of the temporal aggregation of another $GARCH$ process with q' ($q' \geq q$). So, according to Figure 11, the temporal aggregation of a $GARCH$ process induces a decrease in the dependence degree of new series. However, this is not what was verified for market data. Altough non-Guassianity tends to diminish as the time horizon increases, the dependence degree remains the same for lags at least up to 100 days [48]. This result suggests that the memory mechanism present in financial markets is more robust than the multiplicative noise memory mechanism introduced in volatility equation (3), pointing out that amendments in volatility expression are needed to correct this flaw.

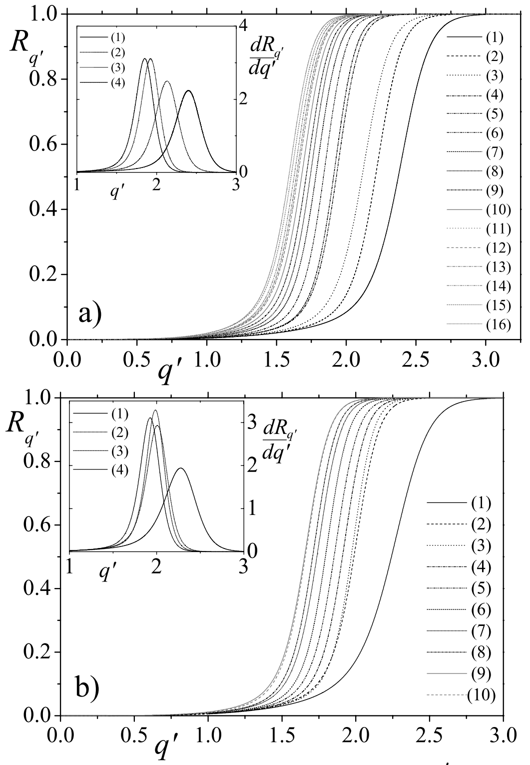

Fig. 10. Dependence criterion $R_{q'}$ vs. q' for various $GARCH(1,1)$ process with typical (q_n, b, c) triplets. In (a) $q_n = 1$ and (b, c) have been chosen as follows: 1-(0, 0), 2-(0.05,0), 3-(0.1,0.2), 4-(0.15,0), 5-(0.2,0), 6-(0.2,0.2), 7-(0.25,0), 8-(0.3, 0), 9-(0.4, 0), 10-(0.4, 0.1), 11-(0.4, 0.2), 12-(0.4, 0.4), 13-(0.5, 0), 14-(0.2, 0.688), 15-(0.35, 0), 16-(0.1, 0). In (b) $q_n = 1.2$ and (b, c) have been chosen as follows: 1-(0, 0), 2-(0.1, 0), 3-(0.1, 0.1), 4-(0.15, 0), 5-(0.2, 0), 6-(0.25, 0), 7-(0.3, 0.1), 8-(0.377, 0), 9-(0.3, 0.45), 10-(0.48, 0.0). The insets contain the derivative $dR_{q'}/dq'$(numerically obtained) for the first four curves as mere illustration.

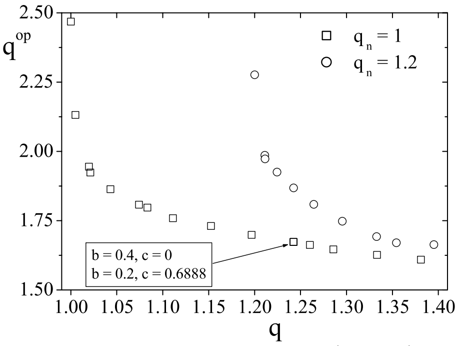

Fig. 11. Plot of q^{op} vs. q for typical (q_n, b, c) triplets. The arrow points two examples which were obtained from *different* triplets, and nevertheless coincide in what concerns the resulting point (q, q^{op}).

5 Concluding remarks

In this article, we have presented a study of the stationary statistical properties of the $GARCH(1, 1)$ stochastic process, which is under specific conditions is equivalent to an infinite linear $ARCH$ process and that is characterised by the exhibition of a time dependent (and exponentially correlated) volatility. This attribute makes this system similar to those presenting fluctuations in some intensive parameter (e.g. temperature), which is at the basis of superstatistics. Inspired by the intimate connection between superstatistics and nonextensive statistical mechanics, we have obtained an expression linking the dynamical parameters (b, c) and the noise nature, q_n, with the entropic index q characterising the stationary distribution for the associated $GARCH(1, 1)$ process. After numerically testing the validity of the approach for the return distributions, we have derived analytical expressions for the squared volatility stationary distribution in the presence of q_n-Gaussian noise. The results are satisfactory for $q_n \simeq 1$ and compatible with the multiplicative noise structure of squared volatility recurrence equations in the same way as models enclosed within the Heston class of financial models. Then, using the q-generalised Kullback-Leibler relative entropy, we have quantified the degree of dependence between successive returns. This analysis led to an entropic index q^{op} which is optimal in the sense that the ratio (33) exhibits maximal sensitivity. We then have verified the existence of a direct relation between q^{op}, the non-Gaussianity, q, and the nature of the noise, q_n. An interesting property emerged, namely that, whatever be the pair (b, c) that results in a certain q for the stationary distribution, one obtains the same value of q^{op}. Consequently, the time series will present the same degree of dependence. Thinking of a GARCH process with smaller q then another process with index $q\prime$ as the convolution for a certain aggregation time of the latter, we have verified that GARCH process does not reproduce the constancy in the dependence degree previously verified in empirical analysis of financial markets. The connection between various entropic indices remind us of the dynamical scenario within which nonextensive statistical mechanics is formulated. Indeed, various entropic indices emerge therein, which coalesce onto the same value $q = 1$ if ergodicity is satisfied. In the present context, this connection can be analysed as follows. Due to the dynamical characteristics of $GARCH(1, 1)$, non-Gaussian distribution for returns $(q \neq 1)$ comes from temporal dependence on its second-order moment which is a self-correlated variable. This correlation, sign of memory in the process, is the responsible for the breakdown of independence [22] between z_t and z_{t+1}, which in turn reflects on the entropic index q^{op}. The connection between q and q^{op} opens the door to the establishment of a relation between the topologies of phase space and probability space. A careful analysis of other kind of systems (e.g. long-range Hamiltonian models, Langevin-like dynamics obeying generalised correlated Fokker-Plank equations) should give a deep insight onto this question.

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